Why Linear Interpolation?

Andrzej Pownuk and Vladik Kreinovich
Computational Science Program
University of Texas at El Paso
500 W. University, El Paso, TX 79968, USA
ampownuk@utep.edu, vladik@utep.edu

Linear interpolation is the computationally simplest of all possible interpolation techniques. Interestingly, it works reasonably well in many practical situations, even in situations when the corresponding computational models are rather complex. In this talk, we explain this empirical fact by showing that linear interpolation is the only interpolation procedure that satisfies several reasonable properties such as consistency and scale-invariance.

By an *interpolation function*, we mean a function $I(x_1, y_1, x_2, y_2, x)$ which is defined for all $x_1 < x < x_2$ and which has the following properties:

- conservativeness:
$$\min(y_1, y_2) \le I(x_1, y_1, x_2, y_2, x) \le \max(y_1, y_2)$$
for all x_i, y_i, and x;
- x-scale-invariance: $I(a \cdot x_1 + b, y_1, a \cdot x_2 + b, y_2, a \cdot x + b) = I(x_1, y_1, x_2, y_2, x)$ for all x_i, y_i, x, $a > 0$, and b;
- y-scale invariance: $I(x_1, a \cdot y_1 + b, x_2, a \cdot y_2 + b, x) = a \cdot I(x_1, y_1, x_2, y_2, x) + b$ for all x_i, y_i, x, $a > 0$, and b;
- consistency:
$$I(x_1, y_1, x_2, y_2, x) = I(x_1', I(x_1, y_1, x_2, y_2, x_1'), x_2', I(x_1, y_1, x_2, y_2, x_2'), x)$$
for all x_i, x_i', y_i, and x; and
- continuity: the expression $I(x_1, y_1, x_2, y_2, x)$ is a continuous function of x, $I(x_1, y_1, x_2, y_2, x) \to y_1$ when $x \to x_1$ and $I(x_1, y_1, x_2, y_2, x) \to y_2$ when $x \to x_2$.

Our main result is that the only interpolation function satisfying all these properties is the linear interpolation

$$I(x_1, y_1, x_2, y_2, x) = \frac{x - x_1}{x_2 - x_1} \cdot y_1 + \frac{x_2 - x}{x_2 - x_1} \cdot y_1.$$