XXXII SYMPOSIUM
ON MATHEMATICAL PHYSICS

Calculation of reliability of structures using fuzzy sets theory

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Basic assumptions

A finite support **random set** on U is a pair (Ξ, m) where Ξ is a finite family of distinct non-empty subsets of U and m is a mapping $\Xi \rightarrow [0,1]$ such that $\sum_{A \in \Xi} m(A) = 1$.

Ξ is called the support of the random set and m a **basic probability assignment**. Each set $A \in \Xi$ contains the possible values of variable x, and $m(A)$ can be viewed as the probability that A is the actual range of x.

Given a random set (Ξ, m), a **belief function** *Bel* can be defined as the following set function

$$\forall A \subseteq U, \quad Bel(\mathrm{A}) = \sum \{m(B) : B \subseteq A, \mathrm{B} \in \Xi\} \qquad (1)$$

plausibility function defined by

$$\forall \mathrm{A} \subseteq \mathrm{U}, \quad Pl(A) = \sum \{m(B) : B \cap A \neq \emptyset, \mathrm{B} \in \Xi\} \qquad (2)$$

It can be shown that

$$Pl(A) = 1 - Bel(\overline{A}) \qquad (3)$$

When Ξ contain only singletons the *Bel*=*Pl* is a **probability measure** (with finite support).

When Ξ is a nested family $A_1 \subseteq A_2 \subseteq ... \subseteq A_p$, then *Bel* and *Pl* satisfy the **decomposability properties**:

$$Bel(A \cap B) = min\{Bel(A), Bel(B)\} \qquad (4)$$
$$Pl(A \cup B) = max\{Pl(A), Pl(B)\} \qquad (5)$$

Relations between random and fuzzy sets

$$(\Xi, m) \Rightarrow \mu_F$$

Fuzzy set F can be defined from any random set (Ξ, m) as follows:

$$\mu_F(u) = \sum_{u \in A} m(A) = Pl(\{u\}) \tag{6}$$

$$\mu_F \Rightarrow (\Xi, m) \tag{7}$$

Let assume that membership the function μ_F is given. $M(F) = \{\alpha_1, ..., \alpha_p\}$ be the set of membership values such that $\alpha_1 > ... > \alpha_p$. $F(\alpha) = \{u : \mu_F(u) \geq \alpha\}$ be α-level-cut.
Then μ_F is equivalent to the unique consonant random set (Ξ, m) defined by:

$$\Xi = \{F(\alpha_i) : i = 1, ..., p\}, \quad m(F(\alpha_i)) = \alpha_i - \alpha_{i+1} \tag{8}$$

with $\alpha_{p+1} = 0$ by convention.

Upper and lower probability

It can be shown that:

$$Bel(A) \leq P(A) \leq Pl(A) \quad \text{for all } A \in \Sigma \tag{9}$$

This equation provides for the definition of two upper and lower distribution function:

$$F_*(x) = Bel([-\infty, x]), \quad F^*(x) = Pl([-\infty, x]) \tag{10}$$

Probability distribution function satisfies the relation:

$$F_*(x) \le F(x) \le F^*(x) \tag{11}$$

If we know fuzzy set membership function we can write:

$$F_*(x) = inf\{1 - \mu_F(\omega) : \omega > x\} \tag{12}$$

$$F^*(x) = sup\{\mu_F(\omega) : \omega \le x\} \tag{13}$$

Probability of failure

$$P_f = 1 - R = 1 - P\{g(\mathbf{x}) \ge 0\} = P\{g(\mathbf{x}) < 0\} \tag{14}$$

$g(\boldsymbol{x})$ – performance function $g : R^m \to R$.

If we know PDF of random variable g we can write:

$$P_f = P\{g < 0\} = P\{g \in (-\infty, 0)\} = \int_{-\infty}^{0} f_g(g) dg \tag{15}$$

$$P_f^- = Bel((-\infty, 0)) \le P_f \le Pl((-\infty, 0)) = P_f^+ \tag{16}$$

Let in the space R^m is given a random set (Ξ, m) then:

$$P_f{}^- == \sum\{m(A_i) : g(A_i) \subseteq (-\infty, 0), A_i \in \Xi\} \tag{17}$$

$$P_f{}^+ == \sum\{m(A_i) : (-\infty, 0) \cap g(A_i) \ne \emptyset, A_i \in \Xi\} \tag{18}$$

These formulas can be used in the Monte Carlo simulations.

Estimation of upper probability of failure based on the fuzzy arithmetic

We assume that $A_i \subset R^m$ i.e. $A_i = A_i^1 \times \ldots \times A_i^m$.

$$m(A_i) = m\left(A_i^1 \times \ldots \times A_i^m\right) \leq min\left\{m_1\left(A_i^1\right), \ldots, m_m\left(A_i^m\right)\right\} \tag{19}$$

where m_i are **basic probability assignments** in the appropriate subspaces.

When Ξ is a nested family $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_p$, then

$$\begin{aligned} P_f^{\,+} &= \sum \{m(A_i) : (-\infty, 0) \cap g(A_i) \neq \emptyset, A_i \in \Xi\} = \\ &= \sum \left\{m\left(A_i^1 \times \ldots \times A_i^m\right) : (-\infty, 0) \cap g(A_i) \neq \emptyset,\ A_i \in \Xi\right\} \leq \\ &\leq \sum \left\{min\left\{m_1\left(A_i^1\right), \ldots, m_m\left(A_i^m\right)\right\} : (-\infty, 0) \cap g(A_i) \neq \emptyset, A_i \in \Xi\right\} = \\ &= \left\{min\left\{\sum m_1\left(A_i^1\right), \ldots, \sum m_m\left(A_i^m\right)\right\} : (-\infty, 0) \cap g(A_i) \neq \emptyset, A_i \in \Xi\right\} = \\ &= min\{\mu(x'_1), \ldots, \mu(x'_m)\} = \mu\left(x'_1, \ldots, x'_m\right) \end{aligned} \tag{20}$$

where $(x'_1, \ldots, x'_m) \in A_i \in \Xi$ and $g(\mathbf{x}') < 0$. Then

$$\boxed{P_f^+ = 1 - R^- = sup\{\mu(\mathbf{x}) : g(\mathbf{x}) < 0\}} \tag{21}$$

Extension principle

Let in the space R^2 is given a random set (Ξ, m), function $f : R^2 \rightarrow R$ and Ξ is a nested family $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_p$, then:

$$\mu(z) = \sum \{m(A_i) : (x, y) \in A_i, A_i \in \Xi, z = f(x, y)\} \tag{22}$$

$$\mu(z) = sup\{\mu(x, y) : z = f(x, y)\} \tag{23}$$

where

$$\mu(x,y)=\sum\{m(A_i):(x,y)\in A_i, A_i\in\Xi\} \tag{24}$$

but

$$\forall A_i\in\Xi \quad A_i=A_i^x\times A_i^y \tag{25}$$

$$m(A_i)=m(A_i^x\times A_i^y)\le min\{m_x(A_i^x),m_y(A_i^y)\} \tag{26}$$

we define

$$\mu_X(x)=\sum\{m_x(A_i^x): x\in A_i^x\} \tag{27}$$

$$\mu_Y(y)=\sum\{m_y(A_i^y): y\in A_i^y\} \tag{28}$$

if we assume the worst case:

$$\mu(z)=sup\{min\{\mu_X(x),\mu_y(y)\}: z=f(x,y)\} \tag{29}$$

Evaluation of fuzzy constants form uncertain experimental data

Let mechanical system is described by the following equation:

$$\mathbf{y}=\mathbf{f}(\mathbf{x},\mathbf{a}) \tag{30}$$

where $\mathbf{f}: R^p\times R^m\to R^q$ and $\mathbf{a}\in R^m$ is a vector of unknown constants.

From experiments we can obtain the following pairs of uncertain data:

$$(\mathbf{x}_1,\mathbf{y}_1),\ \dots\ ,(\mathbf{x}_n,\mathbf{y}_n) \tag{31}$$

where all $\mathbf{x}_i,\mathbf{y}_i$ are fuzzy numbers. We introduce the following notation

$$(\mathbf{x}_1,...,\mathbf{x}_n)=\vec{\mathbf{x}}, \quad (\mathbf{y}_1,...,\mathbf{y}_n)=\vec{\mathbf{y}} \tag{32}$$

Let $(x_1,...,x_n)=\vec{x}\in\vec{\mathbf{x}}_\alpha$, $(y_1,...,y_n)=\vec{y}\in\vec{\mathbf{y}}_\alpha$ where $\vec{\mathbf{x}}_\alpha,\vec{\mathbf{y}}_\alpha$ are α-level-cuts of uncertain data.
Using last squares method we can obtain constants a_i

$$a_i = a_i(\vec{x},\vec{y}) \tag{33}$$

Extreme value of constant can be calculated form the following formulas:

$$a_{i\alpha}^- = inf\{a_i(\vec{x},\vec{y}) : \vec{x}\in\vec{\mathbf{x}}_\alpha, \vec{y}\in\vec{\mathbf{y}}_\alpha\} \tag{34}$$

$$a_{i\alpha}^+ = sup\{a_i(\vec{x},\vec{y}) : \vec{x}\in\vec{\mathbf{x}}_\alpha, \vec{y}\in\vec{\mathbf{y}}_\alpha\} \tag{35}$$

Using α-level-cuts we can define fuzzy numbers A_i:

$$\mu_{A_i}(a_i) = sup\{\alpha : a_\mathrm{i}\in[a_{i\alpha}^-, a_{i\alpha}^+]\} \tag{36}$$

Remarks
1. Random set

$$\Xi_i = \{[a_{i\alpha_1}^-, a_{1\alpha_1}^+], [a_{i\alpha_2}^-, a_{i\alpha_2}^+], ..., [a_{i\alpha_q}^-, a_{i\alpha_q}^+]\} \tag{37}$$

$$m_i([a_{i\alpha_j}^-, a_{i\alpha_j}^+]) = \alpha_{ij} - \alpha_{ij-1} \tag{38}$$

$$\alpha_{i0} = 0 \tag{39}$$

can be used in Monte-Carlo simulation.

Modelling of linear elastic material

Let us assume that $\varepsilon_1,...,\varepsilon_n$ are strain and $\sigma_1,...,\sigma_n$ are stress given from experiments and we are looking for Young module E which satisfy the following relation:

$$\sigma = E \cdot \varepsilon \tag{40}$$

$$y = ax \tag{41}$$

where

$$y := \sigma, \quad x := \varepsilon, \quad a := E \tag{42}$$

$$J = \sum_{i=1}^{n} (y_i - ax_i)^2 = min \tag{43}$$

Calculation of fuzzy material constants

$$[a_\alpha] = hull\{a(x_1,...,x_n, y_1,...,y_n) : x_i \in [x_{i\alpha}], y_i \in [y_{i\alpha}]\} \tag{44}$$

Algorithm of calculation:

$$\text{If } \frac{\partial a_\alpha}{\partial x_i} \geq 0 \text{ then } a_\alpha^- = a(..., x_{i\alpha}^-, ...),\ a_\alpha^+ = a(..., x_{i\alpha}^+, ...) \tag{45}$$

$$\text{If } \frac{\partial a_\alpha}{\partial x_i} < 0 \text{ then } a_\alpha^- = a(..., x_{i\alpha}^+, ...),\ a_\alpha^+ = a(..., x_{i\alpha}^-, ...) \tag{46}$$

$$\text{If } \frac{\partial a_\alpha}{\partial y_i} \geq 0 \text{ then } a_\alpha^- = a(..., y_{i\alpha}^-, ...),\ a_\alpha^+ = a(..., y_{i\alpha}^+, ...) \tag{47}$$

$$\text{If } \frac{\partial a_\alpha}{\partial y_i} < 0 \text{ then } a_\alpha^- = a(..., y_{i\alpha}^+, ...),\ a_\alpha^+ = a(..., y_{i\alpha}^-, ...) \tag{48}$$

Fuzzy number $\tilde{a}$ we can calculate as follows:

$$\mu(a \mid \tilde{a}) = sup\{\alpha : a \in [a_\alpha]\} \tag{49}$$

Fuzzy experimental data

$$\left\lfloor \sigma_{ij}^{ex} \right\rfloor = \left\lfloor \sigma_{ij}^{-}, \sigma_{ij}^{+} \right\rfloor \quad i = 1,...,n, \quad j = 1,...,m_i \tag{50}$$

n – number of stress which was measured
m_i - number of repetition in measurement of the stress σ_i

$$\mu(\sigma_i) = \sum_{\sigma_i \in [\sigma_{ij}^{-}, \sigma_{ij}^{+}]} p_{ij} \qquad j=1,\ldots,m_i \tag{51}$$

$$p_{ij} = m_i\left(\left[\sigma_{ij}\right]\right) = \frac{1}{m_i} \quad \left(\sum_{j=1}^{m_i} p_{ij} = 1 \quad \text{for } i=1,\ldots,n\right) \tag{52}$$

(Ξ_i, m_i) - random sets which contain experimental data
We assume that

$$\bigcap_{j=1}^{m_i} \left[\sigma_{ij}^{ex}\right] = \emptyset \qquad \mathrm{i}=1,\ldots,n \tag{53}$$

Using fuzzy set (51) we can obtain equivalent random sets $(\hat{\Xi}_i, \hat{m}_i)$ which is consonant. It can be shown that:

$$Pl(A) = \sum \{m(B): B \cap A \neq \emptyset, \mathrm{B} \in \Xi\} = \sum \{\hat{m}(B): B \cap A \neq \emptyset, \mathrm{B} \in \hat{\Xi}\} \tag{54}$$

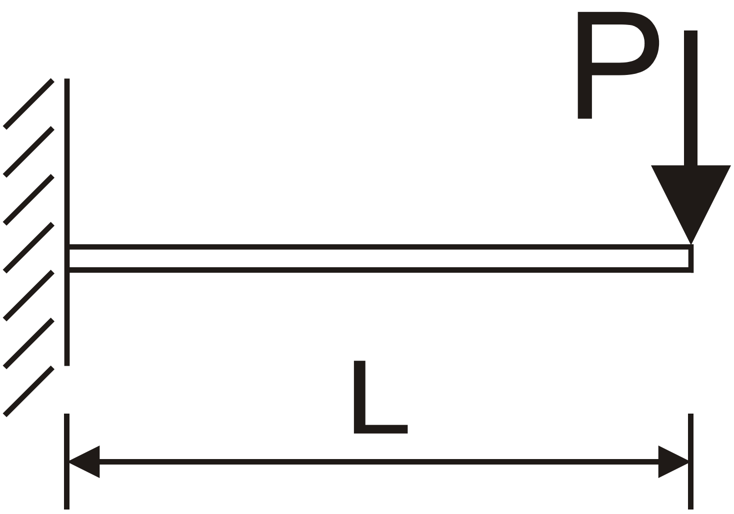
and

$$\mu_F(u) = \sum_{\substack{u \in A \\ A \in \Xi}} m(A) = \sum_{\substack{u \in A \\ A \in \hat{\Xi}}} \hat{m}(A) = Pl(\{u\}) \tag{55}$$

$$P_f^{\,+} = \sum \{m(A_i): (-\infty, 0) \cap g(A_i) \neq \emptyset, A_i \in \Xi\} = \tag{56}$$

$$= \sum \{m(A_i): (-\infty, 0) \cap g(A_i) \neq \emptyset, A_i \in \hat{\Xi}\} \tag{57}$$

Numerical example



Performance function

$$g = u_{max} - u = u_{max} - \frac{PL^3}{3EJ} \tag{58}$$

where $u_{max} = u(L) = 0.01\,m$, $P = 1\,kN$, $E = \tilde{E}\ MPa$, $L = 1\,\mathrm{m}$, $J = \frac{b \cdot h^3}{12}$, $b = 0.02\,m$, $h = 0.045\,m$.

Let

$$\varepsilon_{ij}^{-} = 0.00005 \cdot i + 0.00001 \cdot (j - 3)\ \frac{m}{m} \tag{59}$$

$$\varepsilon_{ij}^{+} = \varepsilon_{ij}^{-} + 0.00005\ \frac{m}{m} \tag{60}$$

$$\sigma_{ij}^{-} = 10 \cdot 10^6 \cdot i + 2 \cdot 10^6 \cdot (j - 3)\ \frac{N}{m^2} \tag{61}$$

$$\sigma_{ij}^{+} = \sigma_{ij}^{-} + 10 \cdot 10^6\ \frac{N}{m^2} \tag{62}$$

Appropriate fuzzy numbers $\tilde{\sigma}_i, \tilde{\varepsilon}_i$ have the following α-level-cuts:

$$[\varepsilon_{i\alpha}] = \left[5\cdot 10^{-5}\cdot i + (\alpha\cdot 5 - 3)\cdot 10^{-5}, 5\cdot 10^{-5}\cdot i + (8 - \alpha\cdot 5)\cdot 10^{-5}\right] \frac{m}{m} \quad (63)$$

$$[\sigma_{i\alpha}] = \left[10\cdot 10^{6}\cdot i + (\alpha\cdot 10 - 6)\cdot 10^{6}, 10\cdot 10^{6}\cdot i + (16 - \alpha\cdot 10)\cdot 10^{6}\right] \frac{N}{m^2} \quad (64)$$

$$\alpha = 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1.0 \quad (65)$$

Using these data we can calculate:

α-level-cuts	$E_\alpha^- \left[10^{11}\frac{N}{m^2}\right]$	$E_\alpha^+ \left[10^{11}\frac{N}{m^2}\right]$
0.2	1.555	2.536
0.4	1.647	2.408
0.6	1.743	2.286
0.8	1.842	2.168
1.0	1.946	2.055

α-level-cuts	$g_\alpha^- \left[10^{-3} m\right]$	$g_\alpha^+ \left[10^{-3} m\right]$
0.2	-1.167	3.151
0.4	-0.005	2.788
0.6	0.003	2.401
0.8	0.057	1.988
1.0	1.076	1.547

Using formula (21) we can calculate:

$$P_f^+ = sup\{\mu(g) : g < 0\} = 0.4 \quad \text{and} \quad R^- = 1 - P_f^+ = 0.6 \quad (66)$$

Conclusions

In this paper a new method of modelling of uncertain parameters was presented. This method can be used instead of Monte-Carlo simulation (based on uncertain data).

This method is based on relation between the theory of random sets and the theory of fuzzy sets.

This method is based on the α-level-cuts of fuzzy numbers because of this it can be used in the fuzzy finite element method and fuzzy boundary element method.

If monotonicity tests can be applied successfully, then this method can be used in modelling of nonlinear problems of computational mechanics.

A new method for calculation of fuzzy material constant is presented. This method is based on uncertain experimental data (intervals or convex sets) and the last squares method. In calculation α-level-cuts and monotonicity test were applied. The method can be extended on the case of fuzzy experimental data.