

Solution of Algebraic Equations by Using Autonomous Computational Methods

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Outline

- 1 Differentiation
- 2 Algebraic Equations
- 3 Machine Learning
- 4 Conclusions and Generalizations

Differentiation

Differentiation

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Input information

- $(f(x) + g(x))' = f'(x) + g'(x)$
- $(f(x) - g(x))' = f'(x) - g'(x)$
- $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$
- $(f(x)/g(x))' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$
- $(f(g(x)))' = f'(g(x))g'(x)$
- arithmetic operations

Differentiation - Sample Application

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Product rule (input information)

$$(f * g)' = f' * g + f * g'$$

After calculations (new theorem created automatically)

$$((f * g) * h)' = (f * g)' * h + (f * g) * h'$$

$$((f * g) * h)' = (f' * g + f * g') * h + (f * g) * h'$$

$$(f * g * h)' = (f' * g) * h + (f * g') * h + (f * g) * h'$$

New theorem can be used in exactly the same way like the original theorem.

$$(f * g * h)' = f' * g * h + f * g' * h + f * g * h'$$

Differentiation (Latex source)

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```
4335 \begin{equation}\cos\left(x\right)+x+2+\sin\left(2\right)\end{equation}
4336 \begin{equation}\cos\left(x\right)+x+2+\sin\left(x\right)\end{equation}
4337 \begin{equation}\cos\left(x\right)+x+2+\sin\left(\sin\left(x\right)\right)\end{equation}
4338 \begin{equation}\cos\left(x\right)+x+2+\sin\left(\cos\left(x\right)\right)\end{equation}
4339 \begin{equation}\cos\left(x\right)+x+2+\cos\left(2\right)\end{equation}
4340 \begin{equation}\cos\left(x\right)+x+2+\cos\left(x\right)\end{equation}
4341 \begin{equation}\cos\left(x\right)+x+2+\cos\left(\sin\left(x\right)\right)\end{equation}
4342 \begin{equation}\cos\left(x\right)+x+2+\cos\left(\cos\left(x\right)\right)\end{equation}
4343 \begin{equation}\cos\left(x\right)+x+2+2\end{equation}
4344 \begin{equation}\cos\left(x\right)+x+2+x\end{equation}
4345 \begin{equation}\cos\left(x\right)+x+2+\sin\left(x\right)\end{equation}
4346 \begin{equation}\cos\left(x\right)+x+2+\cos\left(x\right)\end{equation}
4347 \begin{equation}\cos\left(x\right)+x+2+x\end{equation}
4348 \begin{equation}\cos\left(x\right)+x+2+x\sin\left(x\right)\end{equation}
4349 \begin{equation}\cos\left(x\right)+x+2+x\cos\left(x\right)\end{equation}
4350 \begin{equation}\cos\left(x\right)+x+2+\sin\left(x\right)+\sin\left(x\right)\end{equation}
4351 \begin{equation}\cos\left(x\right)+x+2+\sin\left(x\right)+\cos\left(x\right)\end{equation}
4352 \begin{equation}\cos\left(x\right)+x+2+\cos\left(x\right)+\cos\left(x\right)\end{equation}
4353 \begin{equation}\cos\left(x\right)+x+x\end{equation}
```

Differentiation (step 1)

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$$\cos(x) + (\cos(x) + \cos(2)) \quad (765)$$

$$\cos(x) + (\cos(x) + \cos(x)) \quad (766)$$

$$\cos(x) + (\cos(x) + \cos(\sin(x))) \quad (767)$$

$$\cos(x) + (\cos(x) + \cos(\cos(x))) \quad (768)$$

$$\cos(x) + (\cos(x) + (2 + 2)) \quad (769)$$

$$\cos(x) + (\cos(x) + (2 + x)) \quad (770)$$

$$\cos(x) + (\cos(x) + (2 + \sin(x))) \quad (771)$$

$$\cos(x) + (\cos(x) + (2 + \cos(x))) \quad (772)$$

$$\cos(x) + (\cos(x) + (x + x)) \quad (773)$$

Differentiation (step 2)

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$$\frac{d}{dx} \cos(x) + \frac{d}{dx} (\cos(x) + \cos(2)) \quad (765)$$

$$\frac{d}{dx} \cos(x) + \frac{d}{dx} (\cos(x) + \cos(x)) \quad (766)$$

$$\frac{d}{dx} \cos(x) + \frac{d}{dx} (\cos(x) + \cos(\sin(x))) \quad (767)$$

$$\frac{d}{dx} \cos(x) + \frac{d}{dx} (\cos(x) + \cos(\cos(x))) \quad (768)$$

$$\frac{d}{dx} \cos(x) + \frac{d}{dx} (\cos(x) + (2 + 2)) \quad (769)$$

$$\frac{d}{dx} \cos(x) + \frac{d}{dx} (\cos(x) + (2 + x)) \quad (770)$$

$$\frac{d}{dx} \cos(x) + \frac{d}{dx} (\cos(x) + (2 + \sin(x))) \quad (771)$$

Differentiation (step 3)

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$$(-1) \cdot \sin(x) + \left(\frac{d}{dx} \cos(x) + \frac{d}{dx} \sin(x) \right) \quad (762)$$

$$(-1) \cdot \sin(x) + \left(\frac{d}{dx} \cos(x) + \frac{d}{dx} \sin(\sin(x)) \right) \quad (763)$$

$$(-1) \cdot \sin(x) + \left(\frac{d}{dx} \cos(x) + \frac{d}{dx} \sin(\cos(x)) \right) \quad (764)$$

$$(-1) \cdot \sin(x) + \left(\frac{d}{dx} \cos(x) + \frac{d}{dx} \cos(2) \right) \quad (765)$$

$$(-1) \cdot \sin(x) + \left(\frac{d}{dx} \cos(x) + \frac{d}{dx} \cos(x) \right) \quad (766)$$

$$(-1) \cdot \sin(x) + \left(\frac{d}{dx} \cos(x) + \frac{d}{dx} \cos(\sin(x)) \right) \quad (767)$$

Differentiation (step 4)

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$$(-1) \cdot \sin(x) + \left((-1) \cdot \sin(x) + \left(\frac{d}{du} \cos(u) \right)_{u=\sin(x)} \cdot \frac{d}{dx} \sin(x) \right) \quad (767)$$

$$(-1) \cdot \sin(x) + \left((-1) \cdot \sin(x) + \left(\frac{d}{du} \cos(u) \right)_{u=\cos(x)} \cdot \frac{d}{dx} \cos(x) \right) \quad (768)$$

$$(-1) \cdot \sin(x) + \left((-1) \cdot \sin(x) + \left(\frac{d}{dx} 2 + \frac{d}{dx} 2 \right) \right) \quad (769)$$

$$(-1) \cdot \sin(x) + \left((-1) \cdot \sin(x) + \left(\frac{d}{dx} 2 + \frac{d}{dx} x \right) \right) \quad (770)$$

$$(-1) \cdot \sin(x) + \left((-1) \cdot \sin(x) + \left(\frac{d}{dx} 2 + \frac{d}{dx} \sin(x) \right) \right) \quad (771)$$

$$(-1) \cdot \sin(x) + \left((-1) \cdot \sin(x) + \left(\frac{d}{dx} 2 + \frac{d}{dx} \cos(x) \right) \right) \quad (772)$$

$$(-1) \cdot \sin(x) + \left((-1) \cdot \sin(x) + \left(\frac{d}{dx} x + \frac{d}{dx} x \right) \right) \quad (773)$$

Differentiation (step 5)

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izations

$$(-1) \cdot \sin(x) + ((-1) \cdot \sin(x) + \cos(\sin(x)) \cdot \cos(x)) \quad (763)$$

$$(-1) \cdot \sin(x) + ((-1) \cdot \sin(x) + \cos(\cos(x)) \cdot (-1) \cdot \sin(x)) \quad (764)$$

$$(-1) \cdot \sin(x) + ((-1) \cdot \sin(x) + 0 \cdot 0) \quad (765)$$

$$(-1) \cdot \sin(x) + ((-1) \cdot \sin(x) + (-1) \cdot \sin(x)) \quad (766)$$

$$(-1) \cdot \sin(x) + ((-1) \cdot \sin(x) + (-1) \cdot \sin(\sin(x)) \cdot \cos(x)) \quad (767)$$

$$(-1) \cdot \sin(x) + ((-1) \cdot \sin(x) + (-1) \cdot \sin(\cos(x)) \cdot (-1) \cdot \sin(x)) \quad (768)$$

$$(-1) \cdot \sin(x) + ((-1) \cdot \sin(x) + (0 + 0)) \quad (769)$$

$$(-1) \cdot \sin(x) + ((-1) \cdot \sin(x) + (0 + 1)) \quad (770)$$

$$(-1) \cdot \sin(x) + ((-1) \cdot \sin(x) + (0 + \cos(x))) \quad (771)$$

$$(-1) \cdot \sin(x) + ((-1) \cdot \sin(x) + (0 + (-1) \cdot \sin(x))) \quad (772)$$

$$(-1) \cdot \sin(x) + ((-1) \cdot \sin(x) + (1 + 1)) \quad (773)$$

Group Axioms

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A group is a set (mathematics), G together with an Binary operation \cdot (called the group law of G) that combines any two element elements a and b to form another element, denoted $a \cdot b$. To qualify as a group, the set and operation, (G, \cdot) , must satisfy four requirements known as the group axioms:

- 1 Closure: For all $a, b \in G$, the result of the operation, $a \cdot b$, is also in G .
- 2 Associativity: For all a, b and c in G , $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- 3 Identity element: There exists an element e in G such that, for every element a in G , the equation $1 = e \cdot a = a \cdot e = a$ holds. Such an element is unique.
- 4 Inverse element: For each $a \in G$, there exists an element $b \in G$, commonly denoted a^{-1} (or $-a$, if the operation is denoted $+$, such that $a \cdot b = b \cdot a = e$, where e is the identity element.

Expression Evaluation

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step-1

$$(((1 + 2) + ((1 + 2) - (3 + 4))) + (5 + (6 + 7))) = ((18 - 1) + 0)$$

step-2

$$(((1 + 2) + ((1 + 2) - (3 + 4))) + (5 + (6 + 7))) = (17 + 0)$$

step-3

$$((3 + ((1 + 2) - (3 + 4))) + (5 + (6 + 7))) = ((18 - 1) + 0)$$

step-4

$$(((1 + 2) + ((1 + 2) - (3 + 4))) + (5 + 13)) = ((18 - 1) + 0)$$

step-5

$$(((1 + 2) + (3 - (3 + 4))) + (5 + (6 + 7))) = ((18 - 1) + 0)$$

step-6

$$(((1 + 2) + ((1 + 2) - 7)) + (5 + (6 + 7))) = ((18 - 1) + 0)$$

step-7

$$(((1 + 2) + ((1 + 2) - (3 + 4))) + (5 + (6 + 7))) = 17$$

step-8

$$((3 + ((1 + 2) - (3 + 4))) + (5 + (6 + 7))) = (17 + 0)$$

step-9

$$(((1 + 2) + ((1 + 2) - (3 + 4))) + (5 + 13)) = (17 + 0)$$

step-10

$$(((1 + 2) + (3 - (3 + 4))) + (5 + (6 + 7))) = (17 + 0)$$

Expression Evaluation

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step-93
 $((3 + (-4)) + 18) = (17 + 0)$
step-94
 $((-1) + (5 + 13)) = (17 + 0)$
step-95
 $((-1) + 18) = ((18 - 1) + 0)$
step-96
 $((3 + (-4)) + 18) = 17$
step-97
 $((-1) + (5 + 13)) = 17$
step-98
 $((-1) + 18) = (17 + 0)$
step-99
 $17 = ((18 - 1) + 0)$
step-100
 $((-1) + 18) = 17$
step-101
 $17 = (17 + 0)$
step-102
 $17 = 17$

Expression evaluation is possible without specifying any explicit method for expression evaluation.

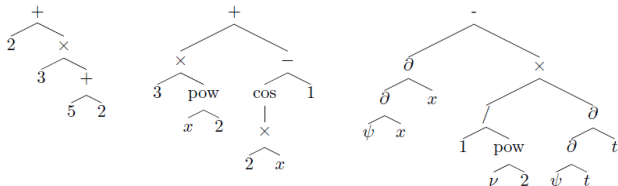
Computational Graph

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COCONUT Project

Continuous CONstraints - Updating the Technology.

<https://www.mat.univie.ac.at/neum/glopt/coconut/>

Solution of Algebraic Equations

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step-1

$$x = (x + (2 + x))$$

step-3

$$x = (x + 2 + x)$$

step-4

$$x = (x + 2 + x)$$

step-5

$$x = x + 2 + x$$

step-6

$$x = x + 2 + x$$

step-7

$$x = 2 + 2 * x$$

step-9

$$x + (-1) * x + (-1) * x = 2$$

Solution of Algebraic Equations

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step-10

$$x + (-2) * x = 2$$

step-11

$$(-1) * x = 2$$

step-12

$$x = (2/(-1))$$

step-13

$$x = (-2)$$

step-14

$$x = ((-2)/1)$$

Automatically Generated Latex Report

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```
step-1\\  
$x=(x+(2+x))$  
\newline  
step-3\\  
$x=(x+2+x)$  
\newline  
step-4\\  
$x=(x+2+x)$  
\newline  
step-5\\  
$x=x+2+x$  
\newline  
step-6\\  
$x=x+2+x$  
\newline  
step-7\\  
$x=2+2*x$  
\newline  
step-9\\  
$x+(-1)*x+(-1)*x=2$
```

Automatically Generated PDF Reports

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Hundred/thousand/unlimited number of examples:

51810 [data-node-126.pdf](#)
52690 [data-node-127.pdf](#)
52354 [data-node-128.pdf](#)
52785 [data-node-130.pdf](#)
52603 [data-node-131.pdf](#)
52512 [data-node-132.pdf](#)
52392 [data-node-134.pdf](#)
52565 [data-node-135.pdf](#)
53310 [data-node-136.pdf](#)
50225 [data-node-14.pdf](#)
52728 [data-node-142.pdf](#)
51848 [data-node-143.pdf](#)
52303 [data-node-144.pdf](#)
52666 [data-node-146.pdf](#)
52696 [data-node-147.pdf](#)
52484 [data-node-148.pdf](#)
50286 [data-node-15.pdf](#)
52539 [data-node-150.pdf](#)

<http://andrew.pownuk.com/research/AlgebraicEquations/>

Simplification of the Solution

Differentiation

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step-1

$$(4 + 6) = (x + x)$$

step-3

$$4 + 6 = (x + x)$$

step-7

$$4 + 6 = x + x$$

step-11

$$4 + 6 = 2 * x$$

step-15

$$4 + 6 + (-2) * x = 0$$

step-21

$$(-2) * x = 0 + (-1) * 4 + (-1) * 6$$

step-24

$$(-2) * x = (-10)$$

step-25

$$x = ((-10)/(-2))$$

step-26

$$x = 5$$

Simplification of the Solution

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- Finding optimal form of the computational process is one or the main goal of this project.
- Quality of the simplification depend on the amount of knowledge available in the system and processing power.
- For small problem the simplification can be found uniquely.
- For more complex problem simplification of the expressions never stops.
- It is possible to use new computational results in order future calculations (self-adaptivity). In this way the system can generate knowledge in autonomous way.

Finite Number of Steps

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- Limits $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$
- Convergence of infinite series
 $\sum_{n=1}^{\infty} \frac{1}{n^p}$ ($p = 2 > 1$ series converges).
- If function f is integrable, then the sequence of Riemann sum $\sum_{n=1}^N f(x_n) \Delta x_n$ converges to $\int_a^b f(x) dx$ (for appropriate partitions).

Presented methodology can be applied to many scientific theories (mathematics, engineering, chemistry, biology etc.) with finite number of steps.

What is Special About This Research?

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All calculations are done in fully **autonomous** way.

Why autonomous calculations?

What is Special About This Research?

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All calculations are done in fully **autonomous** way.

Why autonomous calculations?

Correctness
and
Scalability.

What is Special About This Research?

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All calculations are done in fully **autonomous** way.

Why autonomous calculations?

Results are **NOT Biased**
and **Subjective** .

Classification problem

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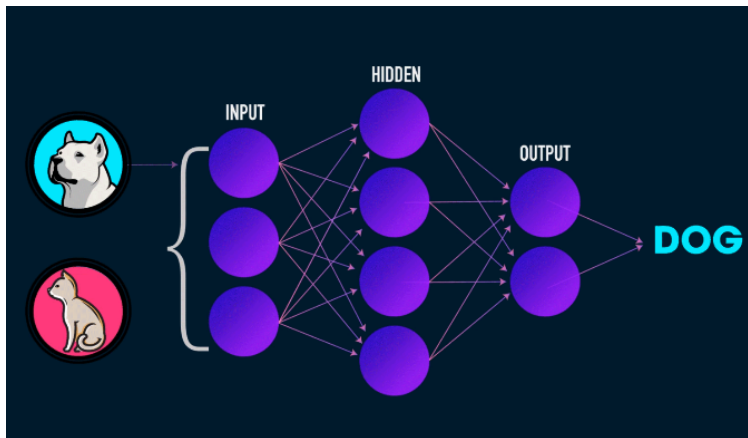


Figure: Classification of pictures

Classification problem from mathematical point of view

Differentiation

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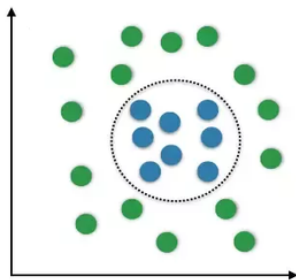
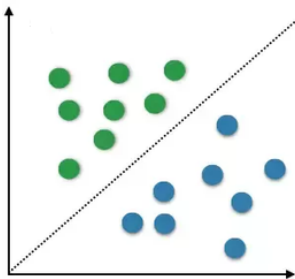
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Example

If x_1 is in the class A then $f(x_1, W) > 0$.

If x_2 is in the class B then $f(x_2, W) < 0$.



Training of the Model

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Least square error:

$$J(W) = \sum_{i=1}^n \sum_{j=1}^{n_o} \left(y_j^{(i)} - \hat{y}_j^{(i)} \right)^2$$

Learning process

$$W^* = \arg \min J(W) = \arg \min \sum_{i=1}^n \sum_{j=1}^{n_o} \left(y_j^{(i)} - \hat{y}_j^{(i)} \right)^2$$

Prediction

$$y_j = \hat{y}_j(x, W^*)$$

Predicting Solution Method

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Let x is a given mathematical problem, then after the training it is possible to predict the solution step y .

$$y = f(x, W)$$

Google AI system proves over 1200 mathematical theorems, April 26th, 2019.

Kshitij Bansal, Sarah M Loos, Markus N. Rabe, Christian Szegedy, and Stewart Wilcox, HOList: An Environment for Machine Learning of Higher-Order Theorem Proving, 24 May 2019, <https://arxiv.org/pdf/1904.03241.pdf>

Finite Number of Steps? Logic?

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izations

- The fundamental difference between if-else and AI is that if-else models are static deterministic environments and machine learning (ML) algorithms, the primary underpinning of AI, are probabilistic stochastic environments.
- Remark: machine learning (probabilistic) reasoning use mathematics which is based on above described methodology.

Solution of Equations with Uncertain Parameters

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The main objective of my present research is to build an autonomous system for automated development of scientific knowledge. The system will be/is applied to automated development of scientific theory of equations with uncertain parameters. The system will be/is capable to automatically expand new scientific ideas (on the basis of existing background knowledge) as well as to improve itself. System would gather existing knowledge, check it in new selected directions, document the process and results, and save new algorithms generated in the process. Once performed research and generated results will be remembered in the system and possible to use if needed.

Computer Based Research Tools

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Scientists use many methods and techniques to perform their research. Also supporting tools are necessary elements for research performance. Discovery/research supporting tools can be as simple as piece of paper and a pencil, but in most subjects they are much more complicated, and recently except specific apparatus support is mostly delivered by computers and advanced software. The computer based research tools should support mathematicians, scientists, and engineers help them make connections to related fields.

Conclusions

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- Mathematical/scientific knowledge can be treated as independent units that can interact with each-other and create new, possibly useful knowledge.
- Generation of new knowledge can be fully automated and autonomous. No interaction with humans is necessary.
- Development of new knowledge is possible in many different fields (e.g. statistics, engineering, chemistry, biology, computer science etc.).

Conclusions (continued)

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- By using presented methodology it is possible to create complex examples and appropriate computational methods relevant to many areas of mathematics as well as in other areas of science and engineering.
- Scientific results can be created in fully objective way without biased opinions of human researchers.
- By using self-adaptive computational methods it is possible to automatically generate new mathematical theorems without interactions with humans (consequently without human errors).
- Machine learning (the main computational method is NOT based on machine learning) can be used as source of good initial guess for processing mathematical information. Actually there is NO main computational method in the system.

Conclusions (continued)

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- Once the information is available in the system it will NEVER be forgotten and can be used for generation of new mathematical theorems. From that perspective the system can be viewed as self-organizing archive of information.
- Development of this and similar systems should speed up cooperation among scientists around the world (theoretical possibility).

Conclusions (continued)

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- Calculations can be done in distributed way (this option is experimental at this moment). Unlimited number of computers can process simultaneously in order to get the results faster. Calculations do not require existence of any centralized system.
- Turning off some computers slows down the calculation.
- Parallel computing can significantly speed up the calculations (future work).
- Autonomous interaction with external sources of information extends internal database of information and should increase productivity of the system (future work).

Conclusions (continued)

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- Mathematical theorems can be used for the solution of many practical engineering and scientific problems if appropriate domain specific knowledge is available.
- I created some practical examples (civil engineering, oil engineering problems) by using presented system but not in fully autonomous way (work in progress).

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Thank you