

# Design of 2D Elastic Structures with the Interval Parameters

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*Abstract:* In many engineering problems exact information about values of the parameters are not know exactly. One of the simplest methods of modeling uncertainty is based on the interval parameters. In order to check the safety of the structure with the interval parameters it is necessary to calculate the interval limit state function. In this paper efficient methods of calculating interval von Mises stress and displacements is presented. The concept of uncertain limit stated was applied in the designing process.

*Key–Words:* Uncertainty, interval parameters, interval finite element method

## 1 Interval parameters in structural mechanics

In engineering, very often, it is not possible to get precise information about the values of the parameters of the structure [1]. Material parameters, for example Young’s modulus, Poisson’s ration very often are not known exactly because of the lack of detail information about the technology which was involved in the production of the parts of the structures. This is particularly important in the complicated composite materials, geomechanics and wood structures. It is really hard to predict exact values of real world loads, which act on each particular structure. Geometrical parameters like height and thickness are also sometimes difficult to estimate. In such situations it is hard to to get reliable probabilistic characteristics of the structure because it is better to apply imprecise probability [2]. In the simplest case it is possible to apply interval parameters. In order to define the interval parameter  $p$  it is necessary to know only upper  $\bar{p}$  and lower bound  $\underline{p}$  of the parameter  $p$ .

$$p \in [\underline{p}, \bar{p}] = \mathbf{p} \quad (1)$$

Engineering structures usually are described by the system of parameter dependent PDE.

$$A(x, p)u = b(x, p) \quad (2)$$

Solution set of the equation (2) can be defined in the following way.

$$\mathbf{u}(x, \mathbf{p}) = \diamond\{u : A(x, p)u = b(x, p), p \in \mathbf{p}\} \quad (3)$$

where  $\mathbf{u}(x, \mathbf{p})$  is the smallest interval which contain the exact solution set  $u(x, \mathbf{p})$  [3]. Exact solution of

the system of partial differential equations is very difficult to obtain. In practice, it is necessary to replace the system of PDE (2) by the system of parameter dependent algebraic equations. Interval solution can be defined in the following way

$$\mathbf{u}(\mathbf{p}) = \diamond\{u(p) : K(p)u = Q(p), p \in \mathbf{p}\} \quad (4)$$

where  $K$  is the stiffness matrix,  $Q$  is the load vector and  $u$  contain vector of displacements.

One of the simplest methods for the estimation of the solution set  $\mathbf{u}(\mathbf{p})$  is the endpoint combination method [3]. Unfortunately, due to high computational complexity, it is not possible to apply it in engineering practice. However, there are very interesting engineering applications that use this approach [4]. Using this approach it is possible to solve nonlinear elastic-plastic problems [6] as well as composite structures [5].

Another very efficient method is based on the response surface method [7, 8, 9, 10]. In this approach the solution  $u = u(p)$  is approximated by some surface  $u(p) \approx u_{approx}(p)$  then all the claculations are based on the function  $u_{approx}(p)$  which is much simpler than original solutions.

A very important group of methods is based on Rump’s theorem [11]. Using Rump’s theorem it is possible to get the results with guaranteed accuracy. In 2001, Muhanna and Mullen published a fundamental paper in that area [12]. The method uses element by element formulation of FEM equations which significantly reduce overestimation of the results. Later, the method was successfully applied and improved by Rama Rao [13]. Popova and Skalna independently applied Rump’s theorem in order to

investigate systems of equations with the matrices in which coefficients depend linearly on the uncertain parameters [15, 14, 16]. The method was applied in order to analyse truss and frame structures. Zhang applied Rump's method to the 2D problems [17].

A very interesting interval finite element method for truss and frame structures was proposed by Neumaier [18]. The method uses special decomposition of the stiffness matrix matrices of the system ( $K = A^T \cdot D \cdot A$ ). The method is very efficient and produces results (displacements) with high accuracy.

## 2 Modified Gradient Method

From a mathematical point of view the problem of solution of the system of equations with the interval parameters is actually an optimization problem.

$$\underline{u}_i = \begin{cases} \min u_i \\ K(p)u = Q(p) \\ p \in \mathbf{p} \end{cases}, \quad (5)$$

$$\bar{u}_i = \begin{cases} \max u_i \\ K(p)u = Q(p) \\ p \in \mathbf{p} \end{cases} \quad (6)$$

There are many optimization methods [19], which can be applied in order to solve the optimization problems (5). One of the simplest is the gradient method. Using this method in order to find a minimum of the function  $u_i = u_i(p)$  it is necessary to search the solution space in the direction of the gradient. For monotone functions  $u_i = u_i(p)$  the maximum and the minimum can be found by using one iteration step.

$$\text{If } \frac{\partial u_i}{\partial p_j} \geq 0 \text{ then } p_j^{\min,i} = \underline{p}_j, p_j^{\max,i} = \bar{p}_j, \quad (7)$$

$$\text{If } \frac{\partial u_i}{\partial p_j} < 0 \text{ then } p_j^{\min,i} = \bar{p}_j, p_j^{\max,i} = \underline{p}_j, \quad (8)$$

$$\underline{u}_i = u_i(p^{\min,i}), \quad \bar{u}_i = u_i(p^{\max,i}). \quad (9)$$

Extreme values of the function can be calculated by using points from the following list

$$L = \{p^{\min,1}, p^{\min,2}, \dots, p^{\max,m}\} \quad (10)$$

Very often some points appear in the list  $L$  multiple times. It is possible to create a list of unique points  $L^*$ .

$$L^* = \{p^{*1}, p^{*2}, \dots, p^{*n}\} \quad (11)$$

In order to get the extreme value of the solution it is enough to find the solution in the points from the list  $L^*$ .

$$\underline{u}_i = \min\{u(p^*) : p^* \in L^*\}, \quad (12)$$

$$\bar{u}_i = \max\{u(p^*) : p^* \in L^*\} \quad (13)$$

Formulas (13) can be applied also in the case when the function  $u_i = u_i(p)$  is not monotone. In that case the points  $p^{\min,i}$  or  $p^{\max,i}$  are not combinations of endpoints or the interval  $\mathbf{p}$  and can be calculated by using general optimization methods. According to many numerical results [21, 20], in engineering problems the method gives exact results or the accuracy is very good. The method is able to solve large scale engineering problems [21]. Using the presented approach it is also possible to solve nonlinear problems of computational mechanics as well as dynamic problems [23]. It is also possible to write general purpose interval FEM software which is based on the gradient method [24]. If the function is not monotone then it is possible to use general optimization methods.

Appropriate derivatives can be calculated by using direct implicit differentiation. For parameter dependent system of equation  $K(p)u = Q(p)$  derivative  $\frac{du}{dp}$  satisfy the following system of equation

$$K \frac{du}{dp} = \frac{dQ}{dp} - \frac{dK}{dp} u \quad (14)$$

In order to get derivative  $\frac{du}{dp}$  it is also possible to apply adjoint variable method.

## 3 Numerical example

Let us consider 2D elasticity problem which is shown on the Fig. 1. In calculations, 64 rectangular elements

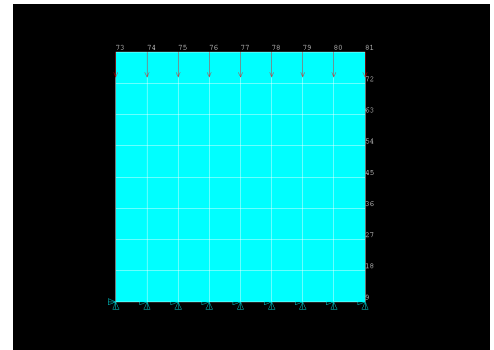


Figure 1: 2D elastic structure

were applied [25]. Example interval  $u_x$  component of displacement was shown on the Fig. 2. Numerical data are the following. Numerical data are the following parameter Young modulus  $E \in [2.0475 \cdot 10^{12}, 2.1525 \cdot 10^{12}]$  [Pa], thickness  $h = 0.1$  [m], Poisson number  $\nu = 0.2$ , point load  $P \in [-1025, -975]$  [N], width 1 [m], height 1 [m]. The problem contains 74 interval parameters. The time of calculations was 72 second on Dell Precision 690 with 3 GHz processor.

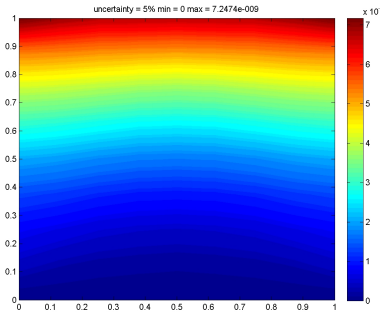


Figure 2: Interval displacement for 5% uncertainty

## 4 Design of Structures with the Interval Parameters

This method allows for the efficient calculations of the interval von Mises stress for different level of uncertainty. These results can be directly applied in the design process. Numerical results are shown below. The dark red area is the potential failure region. The maximum

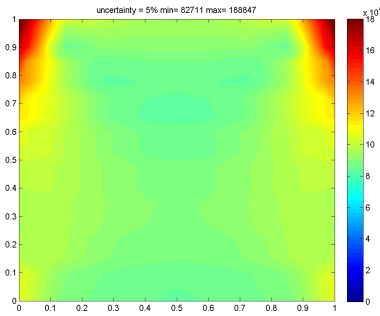


Figure 3: Maximum von Mises for 5% uncertainty

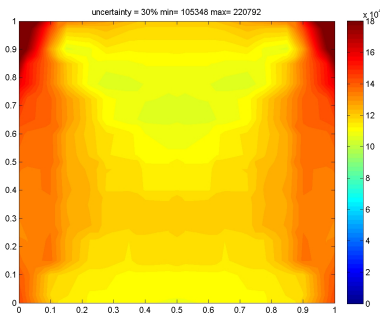


Figure 4: Maximum von Mises for 30% uncertainty

imum von Mises stress is bigger if the uncertainty is bigger. Failure regions are larger if the uncertainty grows. The method generates not only extreme values of the results (e.g. displacements, stress etc.) but

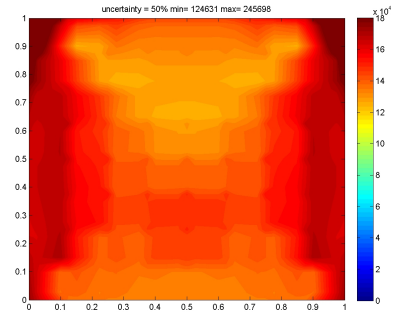


Figure 5: Maximum von Mises for 50% uncertainty

it is also possible to get a combination of parameters which generate each bound of the solution and verify the results using existing engineering software. Example structure in ANSYS is shown on the Fig. 1. Von Mises stress in ANSYS for one combination of parameters is shown on the Fig. 6.

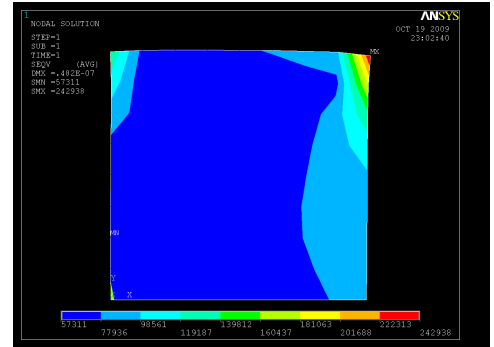


Figure 6: Verification of the results in ANSYS (von Mises stress)

The interval results are symmetric because this is envelop of all possible solutions. The results from ANSYS are not symmetric because they correspond to one specific combination of parameters.

## 5 Accuracy of the Calculations

In order to investigate the accuracy of the calculations it is possible to compare the results of the gradient method with the search methods. In this approach each interval parameter  $\mathbf{p}$  is replaced by a set of grid points  $p \in \{p_1, \dots, p_k\}$ . Extreme values of the solution can be calculated by comparing all possible combinations of the solutions.

$$\underline{u}_i = \min\{u(p^*) : p^* \in \{p_1, \dots, p_k\}\}, \quad (15)$$

$$\bar{u}_i = \max\{u(p^*) : p^* \in \{p_1, \dots, p_k\}\} \quad (16)$$

For 4 element problem with 2% uncertainty, only one bound of the displacement was not calculated exactly and the error of calculation for that one displacement is 0.4%. According to the results of the search method, extreme values of the solution are functions of the endpoints of the intervals, which support the assumption about monotonicity of the solution as a function of the interval parameters.

The method generates not only solution but also appropriate combinations of parameters. Then we can calculate the results using different methods and compare the accuracy.

Let's assume that parameters 1-4 are Young's modulus and 5-7 are point loads. Example, combination of parameters, which correspond to the lower bound of the displacements are shown in the Table below.

**Table 1, Combinations of the parameters which correspond to lower and upper bounds**

$\underline{u}_5$	$\bar{u}_5$
0,1,1,0,0,1,1	1,0,0,1,1,0,0
$\underline{u}_6$	$\bar{u}_6$
0,0,1,1,0,0,0	1,1,0,0,1,1,1
$\underline{u}_7$	$\bar{u}_7$
0,1,0,0,0,0,1	1,0,1,1,1,1,0
$\underline{u}_8$	$\bar{u}_8$
0,1,0,0,0,0,1	1,0,1,1,1,1,0
$\underline{u}_{11}$	$\bar{u}_{11}$
0,1,0,0,0,0,1	1,0,1,1,1,1,0
$\underline{u}_{12}$	$\bar{u}_{12}$
0,1,0,0,0,0,1	1,0,1,1,1,1,0

In Table 1, 0 is a code for lower bound, 1 is a code for upper bound, so for example  $\underline{u}_5$  correspond to 0,1,1,0,0,1,1 which means

$$u_5 = u_5(\underline{E}_1, \bar{E}_2, \bar{E}_3, \underline{E}_4, \underline{P}_1, \bar{P}_2, \bar{P}_3) \quad (17)$$

where  $\underline{E}_1$  is the lower bound of Young's modulus  $E_1$ ,  $E_2$  is the upper bound of Young's modulus  $E_2$  etc. From numerical results we see that the lower bound depend on the  $\underline{E}_i$  (lower bound) then the upper bound depend on  $\bar{E}_i$  (upper bounds). So extreme values of the solution depends on the opposite endpoints of the given intervals. That is one more indicator that the relation  $u_i = u_i(p)$  is very often monotone.

## 6 Conclusions

Using gradient modified gradient method which was presented in this paper it is possible to efficiently design 2D structures with the interval parameters. The method works for linear and non-linear problems of

computational mechanics (however at this moment the method is implemented only for linear problems). The method gives reliable inner bound of the exact solution set. According to some numerical results the methods is exact for simple 2D problems [20]. The accuracy for more complicated problems is very good. However, it is possible to find examples [20] in which the method produces not very accurate results for some component of the solution. In order to detect problems with monotonicity of the solution higher order, monotonicity test can be applied [21].

The method produces not only extreme values of the displacements and stress but also appropriate combinations of parameters which can be used in order to verify the results by using the existing engineering software. The software allows to export models which correspond to given combination of parameters to ANSYS. Example software can be downloaded from the following web page <http://andrzej.pownuk.com>. The method also allows to study different kinds of dependency between different kind of uncertain parameters. In presented example, the maximum von Mises stress are the same for dependent and independent Young Modulus. However, the uncertainty is much bigger in the case of independent Young's modulus.

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