On using global optimization method for approximating interval hull solution of parametric linear systems

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Abstract. Systems of parametric linear interval equations are encountered in many practical applications. Parametric linear interval system is a family of real linear systems. Parametric solution set is a set of all solutions of real systems from the family. In general case the parametric solution set is not an interval vector. Hence, instead of the parametric solution set itself, interval vector containing the solution set (outer interval solution) is calculated. The tightest outer interval solution is called an interval hull solution. To calculate the interval hull solution $2n$ constrained optimization problems are solved using the global optimization method with some accelerating techniques. The monotonicity test is performed using a direct method for solving parametric linear interval systems. Some other techniques like special ordering of subdivided boxes is also used. A bisection and multisection techniques are compared. Various subdivision direction selections rules are tested.

Keywords: parametric linear systems, hull solution, global optimization

1. Introduction

This paper focuses on solving parametric linear systems of structure mechanics with interval parameters. Parametric interval methods allow the engineering practice to account for uncertainty connected either to external factors, such as boundary conditions or applied loads, or to internal factors, such as mechanical or geometric characteristics (Aughenbaugh, 2006; Lallemand, 2000; Muhanna, 2006; Muhanna, 2006; Zalewski et. al., 2006), and to calculate the very sharp bounds on the system response for all possible scenarios in a single analysis (Mullen, 2002).

In general case the parametric solution set is not an interval vector (Neumaier, 1990). Hence, instead of the parametric solution set itself, interval vector containing the parametric solution set (*outer interval solution*) is calculated. The tightest outer interval solution is called an interval hull solution. The problem of computing the hull solution is NP-hard (Rohn and Kreinovich, 1995). However, when the parametric solution is monotone with respect to all interval parameters, interval hull can be calculated by solving at most $2n$ real linear systems.

The problem of calculating hull solution can be written as a problem of solving $2n$ constrained optimization problems. In (Skalna, 2006) an evolutionary optimization methods for approximating (from below) the hull solution has been proposed. One may argue that the underestimation is unknown. However, numerical experiments and the comparison with other methods for solving parametric systems show that the method performs very well.

In this paper global optimization method (GOM for short) with some accelerating techniques is used to calculate the interval hull solution. The monotonicity test is performed using a Direct Method for solving parametric linear interval systems. Some other techniques like special ordering of subdivided boxes are also exploited. A bisection and multisection techniques are compared. Various subdivision direction selections rules are tested.

The paper is organized as follows. The second section contains preliminaries on solving parametric interval linear systems with two disjoint sets of parameters. In the third section, the optimization problem is outlined. This is followed by a description of global optimization algorithm and selected accelerating techniques. Next, some illustrative examples of truss structures and the results of computational experiments are presented. The paper ends with summary conclusions.

2. Preliminaries

Italic faces will be used for real quantities, while bold italic faces will denote their interval counterparts. Let $\mathbb{IR}$ denote a set of real compact intervals $\boldsymbol{x} = [\underline{x}, \overline{x}] = \{x \in \mathbb{R} \mid \underline{x} \leqslant x \leqslant \overline{x}\}$. For two intervals $a, b \in \mathbb{IR}$, $a \geqslant b$, $a \leqslant b$ and $a = b$ will mean that, resp., $\underline{a} \geqslant \overline{b}$, $\overline{a} \geqslant \underline{b}$, and $\underline{a} = \underline{b}$ z $\overline{a} = \overline{b}$. $\mathbb{IR}^n$ will denote interval vectors, $\mathbb{IR}^{n\times n}$ square interval matrices (Neumaier, 1990). The midpoint $\check{x} = \mathrm{m}(\boldsymbol{x}) = (\underline{x} + \overline{x})/2$, the radius $\mathrm{r}(\boldsymbol{x}) = (\overline{x} - \underline{x})/2$, and the width $\mathrm{w}(\boldsymbol{x}) = \overline{x} - \underline{x}$ are applied to interval vectors and matrices componentwise.

Consider linear algebraic system

$$A(p)x(p, q) = b(q) \quad , \tag{1}$$

with linear dependencies

$$a_{ij}(p) = \alpha_{ij0} + \alpha_{ij}^{\mathrm{T}} \cdot p\,, \qquad b_j(q) = \beta_{j0} + \beta_j^{\mathrm{T}} \cdot q \quad , \tag{2}$$

where α_{ij0}, $\beta_{j0} \in \mathbb{R}$, $\alpha_{ij} = \{\alpha_{ij\nu}\} \in \mathbb{R}^k$, $\beta_j = \{\beta_{j\nu}\} \in \mathbb{R}^l$, $i, j = 1, \ldots, n$.

Now assume that some model parameters are unknown. The real vectors p and q are replaced by interval vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ (the real elements are represented by point intervals). This gives a family of the systems

$$A(p)x(p, q) = b(q), \quad p \in \boldsymbol{p},\, q \in \boldsymbol{q} \quad , \tag{3}$$

which is usually written in a symbolic compact form

$$A(\boldsymbol{p})x(\boldsymbol{p}, \boldsymbol{q}) = b(\boldsymbol{q}) \quad , \tag{4}$$

and is called the *parametric interval linear system*. Parametric (*united*) solution set of the system (4) is defined (Jansson, 1991; Kolev, 2004; Rump, 1994) as

$$S(\boldsymbol{p}, \boldsymbol{q}) = \{\, x \mid \exists p \in \boldsymbol{p},\ \exists q \in \boldsymbol{q},\ \ A(p)x(p, q) = b(q)\} \quad . \tag{5}$$

If the solution set $\boldsymbol{S} = S(\boldsymbol{p},\, \boldsymbol{q})$ is bounded, then its interval hull exists and is defined as

$$\square \boldsymbol{S} = [\inf \boldsymbol{S},\, \sup \boldsymbol{S}] = \bigcap \{\boldsymbol{y} \in \mathbb{IR}^n \mid \boldsymbol{S} \subseteq \boldsymbol{y}\} \quad .$$

$\square \boldsymbol{S}$ is called an *interval hull solution.* In order to guarantee that the solution set is bounded, the matrix $A(\boldsymbol{p})$ must be regular, i.e. $A(p)$ must be non-singular for all parameters $p \in \boldsymbol{p}$.

3. Optimization problem

The problem of computing the interval hull solution of the parametric linear system (3) can be written as a problem of solving $2n$ constrained optimization problems

$$\min_{\substack{p \in \boldsymbol{p} \\ q \in \boldsymbol{q}}} x_i(p,\, q),\ i = 1, \ldots, n \tag{6}$$

and

$$\max_{\substack{p \in \boldsymbol{p} \\ q \in \boldsymbol{q}}} x_i(p,\, q),\ i = 1, \ldots, n \tag{7}$$

where $x_i(p,\, q) = \left\{A(p)^{-1}b(q)\right\}_i$ is the i-th coordinate of the solution of the parametric linear system (1), $\boldsymbol{p} \in \mathbb{IR}^k$ and $\boldsymbol{q} \in \mathbb{IR}^l$ are vectors of interval parameters.

Theorem 1. Let $A(\boldsymbol{p})$ be regular, $p \in \mathbb{IR}^k$, and $x^i_{\min}$, $x^i_{\max}$ denote the global solutions of the i-th minimization (6), resp. maximization (7) problems. Then the interval vector

$$\boldsymbol{x} = [x_{\min},\, x_{\max}] = \left(\left[x^i_{\min},\, x^i_{\max}\right]\right)_{i=1}^{n} = \square S(\boldsymbol{p},\, \boldsymbol{q}). \tag{8}$$

The optimization problems (6) and (7) will be solved using a global optimization approach. As a result of the minimization (maximization) problem approximation of the solution set hull, possibly the solution hull itself, will be gained.

4. Global optimization

Global optimization refers to finding the extreme value of a given nonconvex function in a certain feasible region. Solving global optimization problems has made great gain from the interest in the interface between computer science and operations research.

It is assumed in what follows that the inclusion functions have the isotonicity property; i.e., $\boldsymbol{x} \subseteq \boldsymbol{y}$ implies $F(\boldsymbol{x}) \subseteq F(\boldsymbol{y})$ and that for all the inclusion functions holds

$$\mathrm{w}(F(\boldsymbol{x}^i)) \longrightarrow 0 \text{ as } \mathrm{w}(\boldsymbol{x}^i) \longrightarrow 0. \tag{9}$$

4.1. Algorithm

Consider $x(\boldsymbol{p},\ \boldsymbol{q})$, and define $\boldsymbol{r} \in \mathbb{IR}^{k+l}$ with $\boldsymbol{r}_i = \boldsymbol{p}_i$ for $i = 1, \ldots, k$, $\boldsymbol{r}_i = \boldsymbol{q}_i$ for $i = k+1, \ldots, k+l$. Now $x(\boldsymbol{p},\ \boldsymbol{q})$ can be written in shorter form as $x(\boldsymbol{r})$ keeping in mind that x has two vector arguments. Inclusion function is calculated using the Direct Method (Skalna, 2007). It can be easily shown that the method preserves isotonicity property.

The model algorithm is as follows:

Step 0 Set $\boldsymbol{y} = \boldsymbol{r}$ and $f = \min x(\boldsymbol{y})$. Initialize the list $L = \{(f,\ \boldsymbol{y})\}$ and the cutoff level $z = \max x(\boldsymbol{y})$.
Step 1 Choose a coordinate direction using one of the rules: $\nu \in \{1, 2, ..., k+l\}$.
Step 2 Bisect (multisect) $\boldsymbol{y}$ in direction ν: $\boldsymbol{y}_1 \cup \boldsymbol{y}_2$ $\left(\bigcup_{i=1}^{s} \boldsymbol{y}_i,\ \mathrm{int}(\boldsymbol{y}_i) \cap \mathrm{int}(\boldsymbol{y}_j) = \emptyset,\ i \neq j\right)$, int denotes the interior.
Step 3 Calculate $x(\boldsymbol{y}_1)$, $x(\boldsymbol{y}_2)$, and set $f_i = \min x(\boldsymbol{y}_i)$ for $i = 1,\ 2$ and $z = \min\{z,\ \max x(\boldsymbol{y}_1),\ \max x(\boldsymbol{y}_2)\}$.
Step 4 Remove $(f,\ \boldsymbol{y})$ from the list L.
Step 5 Cutoff test: discard the pair $(f_i,\ \boldsymbol{y}_i)$ if $f_i > z$ (where $i \in \{1, 2\}$).
Step 6 Monotonicity test: discard or reduce any remaining pair $(f_i,\ \boldsymbol{y}_i)$ if $0 \notin x_j(\boldsymbol{y}_i)$ for any $j \in \{1, 2, ..., n\}$ and $i = 1, 2$.
Step 7 Add any remaining pairs to the list L. If the list becomes empty, then **STOP**.
Step 8 Denote the pair with the smallest first element by $(f^*, \boldsymbol{y}^*)$.
Step 9 If the width of $x(\boldsymbol{y}^*)$ is less than ε, then print $x(\boldsymbol{y}^*)$ and $\boldsymbol{y}^*$, **STOP**.
Step 10 Go to **Step 1**.

4.2. Midpoint test

The midpoint test is used to reduce the number of intervals in the list L. The pair $(\tilde{f},\ \tilde{\boldsymbol{y}})$ which satisfies $\tilde{f} < f$ for all pairs $(f,\ \boldsymbol{y})$ of the list L is chosen out of L. Then, $\check{f} = \sup F(c)$ is computed, with $c = \mathrm{mid}(\tilde{\boldsymbol{y}})$. Now, all pairs $(f',\ \boldsymbol{y}')$ satisfying $\check{f} < f'$ can be discarded from the list L. Also, a new pair $(f'',\ \boldsymbol{y}'')$ must only be entered in the list L if $\tilde{f} \geqslant f''$ is satisfied.

4.3. Monotonicity test

The monotonicity test is used to figure out whether the function f is strictly monotone in a whole subbox $\boldsymbol{y} \subseteq \boldsymbol{x}$ Then, $\boldsymbol{y}$ cannot contain a global minimizer in its interior. Therefore, if f satisfies

$$\frac{\partial f}{\partial x_i}(\boldsymbol{y}) < 0 \quad \vee \quad \frac{\partial f}{\partial x_i}(\boldsymbol{y}) < 0 \tag{10}$$

then the subbox $\boldsymbol{y}$ can be reduced to one of its edges.

Monotonicity test is pefromed using the Method for Checking the Monotonicity (MCM for short) proposed in (Skalna, 2007). The MCM method is based on a Direct Method (Skalna, 2007) for solving parametric linear systems. Let $f = x(p, q)$. Briefly speaking, the approximations of

$\frac{\partial x}{\partial p_m}(\boldsymbol{p},\, \boldsymbol{q})$, $\frac{\partial x}{\partial q_r}(\boldsymbol{p},\, \boldsymbol{q})$ are obtained by solving the following $k+l$ parametric linear systems

$$A(\boldsymbol{p})\frac{\partial x}{\partial p_m} = b'^m(\boldsymbol{x}^*),\, m = 1,\, \ldots,\, k; \quad A(\boldsymbol{p})\frac{\partial x}{\partial q_r} = b''^r, r = 1,\, \ldots,\, l\ \ , \tag{11}$$

where $b_j'^m(x^*) = -\alpha_{ijm}x_j^*$, $b_j''^r = \beta_{jr}$, $j = 1,\, \ldots,\, n$, $x^* \in \boldsymbol{x}^*$. Detailed description of the MCM method can be found in (Skalna, 2007).

4.4. Subdivision direction selection

Following Ratz and Csendes the interval subdivision direction selection rules has the following merit function:

$$k := \min\left\{j \mid j \in \{1,\, \ldots,\, n\} \text{ and } D(j) = \max_i D(i)\right\} \tag{12}$$

where $D(i)$ is determined by a given rule.

Rule A. The first rule to be applied was the interval-width-oriented rule (Hansen, 1980), it can also be aaplied to non-differentiable function. This rule chooses the coordinate direction with

$$D(i) = w(\boldsymbol{y}). \tag{13}$$

and was justified by the idea that if the original interval is subdivided in a uniform way then the width of the actual subintervals goes to zero most rapidly.

Rule B. Define the indicator

$$p(f_k, f) = \frac{f_k - \underline{f}}{\overline{f} - \underline{f}} \tag{14}$$

that gives which interval is to be selected for subdivision. Here f_k is the approximation of the global minimum value in the iteration k (Casado, 200)

$$f_k = \min\left\{f_l \mid (f_l,\, y_l) \in L\right\}. \tag{15}$$

Rule B selects the coordinate direction for which (12) holds with

$$D(i) = p(f_k,\, f_i). \tag{16}$$

Rule C. Hansen described another rule (initiated by G.W. Walster) (Hansen, 1980). Rule C selects the coordinate direction for which (12) holds with

$$D(i) = \mathrm{w}\left(F_i'(\boldsymbol{y})\mathrm{w}(\boldsymbol{y})\right). \tag{17}$$

4.5. Multisection

Global optimization is based on successive subdivision of the set of feasible solutions. The main idea of multisection technique is to subdivide the problem (in a single step) into many (> 2) smaller problems in contrast to traditional bisection, where to new subintervals are always produced.

5. Examples

To check the performance of the method some illustrative examples of structural mechanical systems are provided. The results of the Global Optimization Method are compared with the results of the Evolutionary Optimization Method (EOM for short) (Skalna, 2006) .

Example 1. (*21-bar plane truss structure*)
For the plane truss structure shown in Fig. 1 the displacements of the nodes are computed. The truss is subjected to downward forces $P_1 = P_2 = P_3 = 30$[kN] as depicted in the figure; Young's modulus $Y = 7.0 \times 10^{10}$[Pa], cross-section area $C = 0.003$[m^2], and length $L = 2$[m]. Assume the stiffness of all bars is uncertain by ±5%. This gives 21 interval parameters.

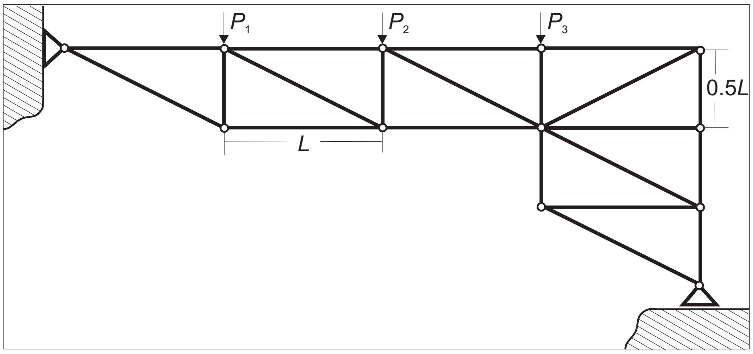

Figure 1. Example 1: 21 planar truss structure

The results produced by the GOM and the EOM methods (Table I) coincide.

Table I. Example 1: results of the GOM and the EOM methods

n	$\underline{x}$ [$\times 10^{-5}$]	$\overline{x}$ [$\times 10^{-5}$]	n	$\underline{x}$ [$\times 10^{-5}$]	$\overline{x}$ [$\times 10^{-5}$]
1	-32.53	-29.27	12	3.90	4.67
2	-1.61	-1.45	13	-16.23	-14.67
3	-26.45	-23.93	14	3.18	3.87
4	-2.41	-2.17	15	-3.63	-2.96
5	-15.78	-14.27	16	3.18	3.87
6	-1.69	-1.37	17	-0.05	0.05
7	-4.08	-3.37	18	2.35	3.02
8	-0.96	-0.57	19	-0.46	-0.40
9	0.36	0.50	20	0.85	1.47
10	3.90	4.67	21	-2.78	-2.09
11	-26.45	-23.93			

Example 2. (Baltimore bridge built in 1870)
Consider the plane truss structure shown in Figure 2 subjected to downward forces of $P_1 = 80[kN]$ at node 11, $P_2 = 120[kN]$ at node 12 and P_1 at node 15; Young's modulus $Y = 2.1 \times 10^{11}$ [Pa], cross-section area $C = 0.004[m^2]$, and length $L = 1$[m]. Assume that the stiffness of 16 bars is uncertain by ±5%. This gives 16 interval parameters.

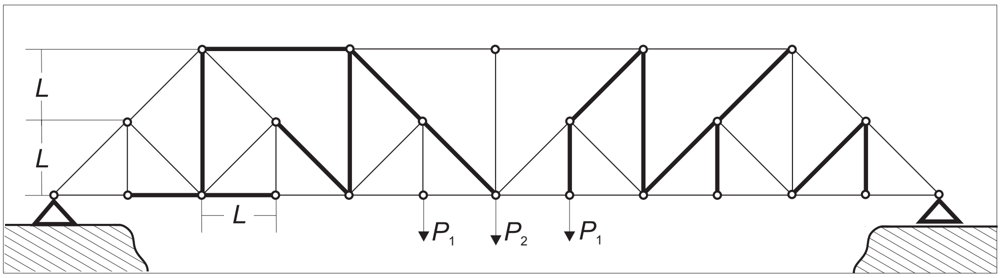

Figure 2. Example 2: Baltimore bridge (built in 1870)

Once again the results of the GOM and the EOM methods coincide. The average relative error produced by both methods equals 2.51%. Maximal relative error equals 34%. For 23 coordinates relative error equals 1%, for another 10 coordinates equals 2%.

6. Conclusions

The problem of solving parametric linear systems has been considered in Section 2. In Section 3 the global optimization method GOM for approximating the solution set hull of parametric linear systems has been described. Computations performed in Section 5 show that the GOM is a powerful tool for solving such systems. The results of the GOM method have been compared with the results of the evolutionary optimization method EOM. Both methods produced the same result which proves that both approaches are powerful tools for solving parametric linear systems. It turns out from the experiments that the monotonicity test and the cutoff test significantly speeds up the convergence of the GOM method, while different rules of subdivision direction selection have no impact on the convergence. Multisection technique is not useful for the problem of solving parametric linear systems since the computation of implicitly given inclusion function is very expensive.

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